

Renormalization of Lorentz non-invariant actions and manifest T-duality

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Synopsis

We study general two-dimensional σ -models which do not possess manifest Lorentz invariance. We show how demanding that Lorentz invariance is recovered as an emergent on-shell symmetry constrains these σ -models. The resulting actions have an underlying group-theoretic structure and resemble Poisson–Lie T-duality invariant actions. We consider the one-loop renormalization of these models and show that the quantum Lorentz anomaly is absent. We calculate the running of the couplings in general and show, with certain non-trivial examples, that this agrees with that of the T-dual models obtained classically from the duality invariant action. Hence, in these cases solving constraints before and after quantization are commuting operations.

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1 Introduction

In theories possessing duality symmetries it is quite standard to lose manifest Lorentz (and even reparameterization) invariance when they are formulated in a duality invariant way at the level of the action. Common characteristic features of these theories are the doubling of fields, the replacement of the Lorentz by some other symmetry as well as the recovery of the usual Lorentz symmetry on-shell (see [1, 2, 3, 4], for notable examples of this). In addition, in such duality invariant actions it is possible to use the equations of

motion to eliminate half of the fields resulting in a Lorentz invariant action. If this can be done in more than one way, the corresponding theories are duality equivalent. There are two issues that immediately arise which are actually related. Firstly, it is important to know whether such an equivalence holds beyond the classical level. Secondly, is the procedure of quantization before eliminating extra fields equivalent to first eliminating fields and then quantizing?

These issues can be addressed successfully with a generalization of the well known abelian [5] and non-abelian [6] T-dualities, namely the so-called Poisson-Lie T-duality [7]. Its most notable feature is that it does not rely on the existence of isometries but rather on a rigid group-theoretical structure [7] known as the Drinfeld Double. Nevertheless, it shares some common features with ordinary T-duality. For instance, it can be explicitly formulated as a canonical transformation between phase-space variables [8, 9], similarly to ordinary T-duality [10, 11].

Pairs of Poisson-Lie T-dual σ -models can be constructed classically from a single duality invariant theory consisting of a WZW model on the Drinfeld Double supplemented with a non-linear interaction term [12]. By choosing a parametrization for a group element of the Double and implementing the constraints that arise as equations of motion, one can eliminate half of the fields and obtain a standard Lorentz invariant σ -model. The T-dual partner to this is obtained by choosing an inequivalent parametrization for the group element and repeating the constraint procedure. A first step in understanding whether Poisson-Lie T-duality holds beyond the classical level was made in [13] where it was shown, for specific examples, that the system of renormalization group (RG) equations for couplings occurring in each of the T-dual theories are equivalent at one-loop. The one-loop renormalization of general Poisson-Lie T-dual models was considered in [14] and a proof of the equivalence of their RG equations was given in [15].

It is interesting to ask whether these RG equations can be recast in a manifestly duality invariant form. For this to be so we should hope that the duality invariant theory also produces the same system of RG equations. This need not be the case since the process of constraining used to arrive at the pair of T-dual models may not commute with the process of quantization. This motivates our study of the one-loop renormalization of the T-duality invariant theory in particular and of Lorentz non-invariant σ -models in general.

The organization of this paper is as follows: In section 2 we consider general two-dimensional σ -models that do not possess Lorentz invariance. We show that demanding

Lorentz invariance as an emergent on-shell symmetry severely constrains the background of the σ -model. In fact, solving these constraints results in an action that has the form of the Poisson–Lie T-duality action described above.

In section 3 we consider the quantum behavior of these Lorentz non-invariant theories by calculating their one-loop effective actions using the background field method. The renormalization of similar theories was considered [16, 17] in the related context of the Doubled Formalism of abelian T-duality [18] (a larger class of such doubled models has been considered at the classical level in [19]). In our case however, we must extend this analysis to account for the non-trivial group geometry of the group manifold. One might expect both a Weyl anomaly and a Lorentz anomaly to occur. We find that the counter term for the Weyl divergences can be absorbed into a redefinition of the coupling constants and hence the models are one-loop renormalizable. Furthermore we show that the potential Lorentz anomaly vanishes through non-trivial cancellations. This indicates quantum consistency of the Lorentz non-invariant theories considered.

In section 4 we apply our general construction to the Poisson–Lie T-duality invariant σ -model i.e. the case when the group manifold is a Drinfeld Double. In this case the explicit expressions we construct for the Weyl anomaly provide us a manifestly duality invariant description of the RG equations derived previously. For specific non-trivial examples we check that these do indeed reproduce the RG equations obtained before [20, 15].

We conclude our paper with section 5.

2 On-shell Lorentz invariant actions

In this section we develop the classical formalism for two-dimensional σ -models to possess on-shell Lorentz invariance. For a related treatment the reader is also referred to [19].

Consider a general Lorentz non-invariant two-dimensional bosonic σ -model action [2]¹

$$S = \frac{1}{2} \int d\sigma d\tau (C_{MN} \partial_0 X^M \partial_1 X^N + M_{MN} \partial_1 X^M \partial_1 X^N) , \quad (2.1)$$

where the general matrix C_{MN} and the symmetric matrix M_{MN} depend on the X^M 's. It is straightforward to show that the Lorentz transformations

$$\delta X^M = -\sigma \partial_\tau X^M - \tau \partial_\sigma X^M , \quad (2.2)$$

¹In our conventions, the light-cone world-sheet variables are $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$ and hence $\partial_\pm = \partial_0 \pm \partial_1$, where $\partial_0 = \partial_\tau$ and $\partial_1 = \partial_\sigma$. The two-dimensional world-sheet metric is diagonal with $\eta_{00} = -\eta_{11} = 1$ and also $\epsilon^{01} = 1$.

do not leave (2.1) invariant, instead they result in

$$\delta_{\text{Lorentz}} S = \int d\sigma d\tau \left(\frac{1}{2} S_{MN} (\partial_1 X^M \partial_1 X^N + \partial_0 X^M \partial_0 X^N) + M_{MN} \partial_0 X^M \partial_1 X^N \right) , \quad (2.3)$$

where S_{MN} is the symmetric part of C_{MN} , i.e. $S_{MN} = \frac{1}{2}(C_{MN} + C_{NM})$. We shall find it convenient to split the index M such that $X^M = (X^\mu, Y^i)$. We would like to find the conditions under which (2.1) is on-shell Lorentz invariant,² using the equations of motion corresponding to the X^μ 's. Since the equations of motion for the Y^i 's will not be used, we may add to the Lagrangian corresponding to (2.1) any Lorentz invariant term $Q_{ij} \partial_+ Y^i \partial_- Y^j$, for some matrix Q_{ij} that may depend on the Y^i 's. The fields Y^i are called spectators. The variation of (2.1) with respect to these fields is

$$\delta S = \int d\sigma d\tau \delta X^\mu \left(-\partial_1 E_\mu + \frac{1}{2} \partial_\mu M_{N\Lambda} \partial_1 X^N \partial_1 X^\Lambda + \hat{\Gamma}_{N;\mu\Lambda} \partial_0 X^N \partial_1 X^\Lambda \right) , \quad (2.4)$$

where

$$E_\mu = S_{\mu N} \partial_0 X^N + M_{\mu N} \partial_1 X^N , \quad (2.5)$$

and

$$\hat{\Gamma}_{N;\mu\Lambda} = \frac{1}{2} (\partial_\mu C_{N\Lambda} + \partial_\Lambda C_{\mu N} - \partial_N C_{\mu\Lambda}) , \quad (2.6)$$

which as we shall soon see, when it is appropriately restricted, plays the rôle of connection.

Since (2.5) and the equations of motions are first and second order in world-sheet derivatives, respectively, we should require that the latter can be written in the form $\partial_1(\dots)$ in order to get conditions that can be used to make (2.3) vanish. However, the last two terms in (2.4) cannot be written immediately in this form. Had this been the case performing the derivative would result into second order derivatives like ∂_1^2 and $\partial_0 \partial_1$ which do not appear in (2.4).

To proceed we require that the last two terms in (2.5) can be cast into the form

$$-\partial_1 \Lambda_A^\nu \Lambda_\mu^A E_\nu , \quad (2.7)$$

for some X^μ dependent square matrix Λ_μ^A and its inverse Λ_A^μ . This is a very stringent condition with severe consequences that restrict the type of backgrounds in the σ -models that can be finally admitted by requiring that Lorentz invariance emerges on-shell. Then, assuming that (2.7) holds, we find that (2.4) becomes

$$\delta S = - \int d\sigma d\tau \delta X^\mu (\partial_1 E_\mu + \partial_1 \Lambda_A^\nu \Lambda_\mu^A E_\nu) = - \int d\sigma d\tau \delta X^\mu \Lambda_\mu^A \partial_1 (\Lambda_A^\nu E_\nu) . \quad (2.8)$$

²Similar considerations have been made in [2] for constant matrices C and M .

Hence, the equations of motion can be integrated once and read

$$\Lambda_A^\nu E_\nu = f_A(\tau) , \quad (2.9)$$

where the $f_A(\tau)$'s are otherwise arbitrary. However (2.8) shows that (2.1) has a “small” local symmetry under the transformation

$$\delta X^\mu = \Lambda_A^\mu \epsilon^A(\tau) , \quad (2.10)$$

where $\epsilon^A(\tau)$ are some τ -dependent parameters. This can be used to set $f^A(\tau) = 0, \forall A$, showing that the equations of motion are first order and simply read

$$E_\mu = 0 , \quad (2.11)$$

where the E_μ 's are given by (2.5).

2.1 Recovering on-shell Lorentz invariance

Using (2.11), we may solve for $\partial_0 X^\mu$ and after some algebraic manipulations recast the anomalous Lorentz variation (2.3) into the form

$$\begin{aligned} \delta_{\text{Lorentz}} S \Big|_{\text{on shell}} &= \int d\sigma d\tau \left[\frac{1}{2} (S_{MN} - M_{M\alpha} S^{\alpha\beta} M_{\beta N}) \partial_1 X^M \partial_1 X^N \right. \\ &\quad \left. + (M_{iN} - S_{i\alpha} S^{\alpha\beta} M_{\beta N}) \partial_0 Y^i \partial_1 X^N + \frac{1}{2} (S_{ij} - S_{i\alpha} S^{\alpha\beta} S_{\beta j}) \partial_0 Y^i \partial_0 Y^j \right] , \end{aligned} \quad (2.12)$$

where $S^{\alpha\beta}$ is the inverse matrix to $S_{\alpha\beta}$. Imposing that this vanishes we get the conditions

$$\begin{aligned} S_{MN} &= M_{M\alpha} S^{\alpha\beta} M_{\beta N} , \\ S_{ij} &= S_{i\alpha} S^{\alpha\beta} S_{\beta j} , \\ M_{iN} &= S_{i\alpha} S^{\alpha\beta} M_{\beta N} . \end{aligned} \quad (2.13)$$

Having established the on-shell Lorentz invariance of the action it remains to show the Lorentz invariance of the equations of motion (2.11). In order to do that we define a set of projection operators as

$$(P_\pm)^\mu{}_\nu = \frac{1}{2} (\delta^\mu{}_\nu \mp S^{\mu\lambda} M_{\lambda\nu}) \quad (2.14)$$

and their invariant subspaces as

$$(Q_\pm)^\mu{}_i = \frac{1}{2} S^{\mu\nu} (S_{\nu i} \mp M_{\lambda i}) . \quad (2.15)$$

Indeed, using (2.13), one can readily verify that they satisfy the required properties

$$P_{\pm}^2 = P_{\pm} , \quad P_{\pm} P_{\mp} = 0 , \quad P_{\pm} Q_{\pm} = Q_{\pm} , \quad P_{\pm} Q_{\mp} = 0 . \quad (2.16)$$

Then (2.11) with the definition (2.5) can be written in the form

$$E_{\mu} = S_{\mu\nu} (E_{+}^{\nu} + E_{-}^{\nu}) , \quad (2.17)$$

where

$$E_{\pm}^{\mu} = (P_{\mp})^{\mu}_{\nu} \partial_{\pm} X^{\nu} + (Q_{\mp})^{\mu}_i \partial_{\pm} Y^i . \quad (2.18)$$

Then using the properties (2.16) the equations of motion (2.11) can be easily shown to be equivalent to

$$E_{\pm}^{\mu} = 0 , \quad (2.19)$$

which has the required form, since they remain invariant under Lorentz transformations.

2.1.1 The off-shell symmetry

In addition to the on-shell Lorentz symmetry (2.2) we can construct some modified Lorentz transformations, similar to those which appear in a simpler setting in [1], under which the action is invariant off-shell. These transform the fields as

$$\begin{aligned} \delta X^{\mu} &= -\sigma \partial_{\tau} X^{\mu} - \tau \partial_{\sigma} X^{\mu} - \sigma (E_{+}^{\mu} + E_{-}^{\mu}) , \\ \delta Y^i &= -\sigma \partial_{\tau} Y^i - \tau \partial_{\sigma} Y^i . \end{aligned} \quad (2.20)$$

We see that the Y^i 's have the usual global Lorentz transformation rules. On-shell the same is of course true for the X^{μ} 's as well.

2.2 Solving the conditions

Let's introduce a set of vielbeins e_{μ}^A and their inverses e_A^{μ} . Then we may write $S_{\mu\nu}$ in the standard form

$$S_{\mu\nu} = e_{\mu}^A e_{\nu}^B \eta_{AB} , \quad (2.21)$$

where η_{AB} is the tangent space metric. Similarly, let's choose matrices M_{MN} such that

$$M_{\mu\nu} = -R_{AB} e_{\mu}^A e_{\nu}^B , \quad (2.22)$$

for some constant symmetric matrix R_{AB} .

Then, the conditions (2.13) are solved by introducing a set of matrices F_i^a that may depend on the spectator field Y^i . We obtain

$$\begin{aligned}
S_{\mu i} &= \eta_{AB} e_\mu^A F_i^B , \\
S_{ij} &= \eta_{AB} F_i^A F_j^B , \\
M_{ij} &= -R_{AB} F_i^A F_j^B . \\
M_{\mu i} &= -R_{AB} e_\mu^A F_i^B .
\end{aligned} \tag{2.23}$$

We may choose for the matrix C_{ij} to equal S_{ij} since its antisymmetric part is Lorentz invariant by itself. In addition, R_{AB} obeys the compatibility condition

$$R_{AC} \eta^{CD} R_{DB} = \eta_{AB} . \tag{2.24}$$

The matrix R_{AB} can be taken to be equal to unity by appropriate choice of the vielbeins. Then (2.24) shows that $\eta^{-1} = \eta$, which implies that η is diagonal with plus and minus entries [2]. Having in mind applications to T-duality we take η to be traceless as well.

To proceed further, we find it necessary to impose for the matrix M_{MN} the conditions

$$\begin{aligned}
\hat{\nabla}_\mu M_{\nu\lambda} &= 0 , \\
\hat{\nabla}_\mu M_{\nu i} &= 0 , \\
\partial_\mu M_{ij} &= 0 ,
\end{aligned} \tag{2.25}$$

for some connection $\Omega_{\nu\lambda}^\mu$. In the covariant differentiation the index i is assumed not to transform. Moreover, we impose that

$$\hat{\Gamma}_{\mu;\nu i} = \hat{\Gamma}_{j;\nu i} = 0 , \quad \partial_i C_{\mu\nu} = 0 , \tag{2.26}$$

Then the last two terms in (2.4) are written as

$$\frac{1}{2} \partial_\mu M_{N\Lambda} \partial_1 X^N \partial_1 X^\Lambda + \hat{\Gamma}_{N;\mu\Lambda} \partial_0 X^N \partial_1 X^\Lambda = \left(\Omega_{\mu\lambda}^\nu M_{\nu\Lambda} \partial_1 X^N + \hat{\Gamma}_{N;\mu\lambda} \partial_0 X^N \right) \partial_1 X^\Lambda . \tag{2.27}$$

Identifying the connection as

$$\Omega_{\nu\lambda}^\mu = S^{\mu\rho} \hat{\Gamma}_{\rho;\nu\lambda} , \tag{2.28}$$

we find that the of right hand side of (2.27) becomes

$$\begin{aligned}
&\left[\hat{\Gamma}_{\mu\lambda}^\nu (M_{\nu N} \partial_1 X^N + S_{\nu\lambda} \partial_0 X^\lambda) + \hat{\Gamma}_{i;\mu\lambda} \partial_0 X^i \right] \partial_1 X^\lambda \\
&= \hat{\Gamma}_{\mu\lambda}^\nu E_\nu \partial_1 X^\lambda + (\hat{\Gamma}_{i;\mu\lambda} - S_{\nu i} \hat{\Gamma}_{\mu\lambda}^\nu) \partial_0 X^i \partial_1 X^\lambda .
\end{aligned} \tag{2.29}$$

From the above and the assumption (2.7), we see that the form of the connection should be

$$\hat{\Gamma}_{\mu\lambda}^\nu = \Lambda_A^\nu \partial_\lambda \Lambda_\mu^A \implies \Gamma_{\lambda\mu}^\nu = \Lambda_A^\nu \partial_\lambda \Lambda_\mu^A - \frac{1}{2} H^\nu{}_{\lambda\mu} , \quad (2.30)$$

in order for the “small” gauge invariance (2.9) to exist. In the second step we have written the expression for the usual symmetric Levi-Civita connection $\Gamma_{\mu\lambda}^\nu$ constructed out of the symmetric tensor $S_{\mu\nu}$. The above expression implies that

$$\partial_\mu \Lambda_\lambda^A - \partial_\lambda \Lambda_\mu^A = \Lambda_\nu^A H^\nu{}_{\mu\lambda} . \quad (2.31)$$

In addition, the last term in (2.29) should vanish, that is

$$\hat{\Gamma}_{i;\mu\lambda} = S_{\nu i} \hat{\Gamma}_{\mu\lambda}^\nu = S_{\nu i} \Lambda_A^\nu \partial_\lambda \Lambda_\mu^A , \quad (2.32)$$

should hold. From (2.26) it is easy to deduce that $C_{\mu i}$ is a total derivative. Hence, with no loss of generality we take

$$C_{\mu i} = 0 . \quad (2.33)$$

Then with the help of (2.26) we write (2.32) as

$$\frac{1}{2} \partial_\mu C_{i\lambda} = S_{\nu i} \Lambda_A^\nu \partial_\lambda \Lambda_\mu^A . \quad (2.34)$$

Finally, note that due to the first of (2.25) and (2.22) the (torsion free) spin connection and the torsion are related as³

$$\omega_\mu{}^{AB} + \frac{1}{2} H_\mu{}^{\nu\lambda} e_\nu^A e_\lambda^B = 0 . \quad (2.36)$$

The form of the condition (2.31) suggests that the background corresponds to a group manifold. With this in mind, we now set up the notation and define the necessary quantities. Let the X^μ 's parametrize an element G of a group D . We introduce a set of representation matrices $\{T^A\}$, $A = 1, 2, \dots, \dim(D)$, satisfying

$$[T^A, T^B] = i f^{AB}{}_C T^C . \quad (2.37)$$

Next we introduce the left-invariant Maurer-Cartan forms

$$L_\mu^A = -i \text{Tr}(T^A G^{-1} \partial_\mu G) , \quad (2.38)$$

³In our conventions

$$\hat{\nabla}_\mu V_\nu^a = \nabla_\mu V_\nu^a - \frac{1}{2} H_{\mu\nu}{}^\lambda V_\lambda^a , \quad \nabla_\mu V_\nu^a = \partial_\mu V_\nu^a - \Gamma_{\mu\nu}^\lambda V_\lambda^a + \omega_\mu{}^a{}_b V_\nu^b \quad (2.35)$$

and $\nabla_\mu e_\nu^a = 0$.

with the fundamental property, the Maurer–Cartan equation,

$$\partial_\mu L_\nu^A - \partial_\nu L_\mu^A = f^A_{BC} L_\mu^B L_\nu^C . \quad (2.39)$$

We also define the right-invariant Maurer–Cartan forms

$$R_\mu^A = -i \text{Tr}(T^A \partial_\mu G G^{-1}) \quad (2.40)$$

for which

$$\partial_\mu R_\nu^A - \partial_\nu R_\mu^A = -f^A_{BC} R_\mu^B R_\nu^C . \quad (2.41)$$

The two forms are related by

$$R_\mu^A = C^A_B L_\mu^B , \quad (2.42)$$

where the matrix C^{AB} (not related to the one in (2.1)) is defined as

$$C^{AB} = \text{Tr}(T^A G T^B G^{-1}) , \quad (2.43)$$

and indices are raised and lowered with the tangent space metric. C^{AB} is orthogonal and among the many properties it obeys we will need that

$$(C^T \partial_\mu C)^{AB} = f^{AB}_C L_\mu^C . \quad (2.44)$$

Using the above, if we choose as our vielbein the left-invariant Maurer–Cartan forms

$$e_\mu^A = L_\mu^A , \quad (2.45)$$

we obtain from (2.39) that the spin connection is

$$\omega_\mu^{AB} = \frac{1}{2} f^{AB}_C L_\mu^C . \quad (2.46)$$

Then from (2.36) we have to choose that

$$H^{AB}_C = -f^{AB}_C , \quad (2.47)$$

in tangent space indices. Also from (2.31) and (2.41), the matrix $\Lambda_\mu^A = R_\mu^A$.

Instead, if we choose as our vielbein the right-invariant Maurer–Cartan forms

$$e_\mu^A = R_\mu^A , \quad (2.48)$$

then from (2.41) we get for the spin connection

$$\omega_\mu^{AB} = -\frac{1}{2} f^{AB}_C R_\mu^C . \quad (2.49)$$

Then from (2.36) we have to choose that

$$H^{AB}{}_C = +f^{AB}{}_C , \quad (2.50)$$

in tangent space indices. Also from (2.31) and (2.39) the matrix $\Lambda_\mu^A = L_\mu^A$.

Of course quantities and relations that can be computed using the metric are frame independent. For instance, the Riemann curvature is

$$R_{AB;CD} = \frac{1}{4} f_{AB}{}^E f_{ECD} \quad (2.51)$$

and we note the identity

$$\nabla_\mu H_{\nu\rho\lambda} = 0 , \quad (2.52)$$

for torsion proportional to the structure constants as in our case. Also in WZW models, the fine tuning of geometry and torsion gives the identity [21]

$$\hat{R}_{\mu\nu;\rho\lambda} = 0 , \quad (2.53)$$

where the generalized Riemann curvature tensor that uses $\hat{\Gamma}_{\nu\lambda}^\mu$ as a connection is defined in (3.8).

In what follows we choose to work with the left-invariant forms as our vielbein's, that is with the choice (2.45).

From (2.26) and (2.44) we find from (2.34) that

$$C_{i\mu} = 2\eta_{AB} F_i^A e_\mu^B , \quad (2.54)$$

where we repeat that the vielbein is given by (2.45).

Gathering the previous scattered results we obtain the action with Lagrangian density

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{\text{WZW}} + \frac{1}{2} \eta_{AB} (F_i^A F_j^B \partial_0 Y^i \partial_1 Y^j + 2F_i^A L_\mu^B \partial_0 Y^i \partial_1 X^\mu) \\ & + \frac{1}{2} R_{AB} (-L_\mu^A L_\nu^B \partial_1 X^\mu \partial_1 X^\nu - F_i^A F_j^B \partial_1 X^i \partial_1 Y^j - 2L_i^A F_i^B \partial_1 X^\mu \partial_1 Y^i) , \end{aligned} \quad (2.55)$$

where \mathcal{L}_{WZW} corresponds to a WZW model action. It is not difficult to check that this has the form of the duality invariant action used for Poisson–Lie T-duality. In particular, one can verify that by comparing with eq. (B.1) of [9]. We emphasize that, unlike the discussions related to Poisson-Lie T-duality, the group D is not necessarily a Drinfeld double, although some important applications of our construction will be in that direction, in section 4.

3 RG flows of Non-Lorentz invariant actions

In this section we shall explain in detail the computation of the effective action of the two-dimensional bosonic σ -model action (2.1) or equivalently of (2.55), since we want to have an emergent Lorentz symmetry on-shell. To simplify the discussion we disregard completely the spectator fields Y^i .

3.1 A brief review of the background field expansion

In order to compute the quantum anomalies of the actions (2.1) and (2.55), both Weyl and Lorentz ones, it is necessary to calculate the effective action. For that purpose we will use the covariant background field method [22, 21] and in particular the algorithmic method of [23] and [24] to study the renormalization of this two-dimensional sigma model. Our starting point is to expand the fields X^μ around the classical solution χ^μ and its fluctuation π^μ as $X^\mu = \chi^\mu + \pi^\mu$. However, this split leads to a non-covariant expansion, since π^μ cannot be interpreted as a vector. To achieve covariance, we use a non-linear split based on the tangent vector ξ^μ of the geodesic that connects χ^μ and $\chi^\mu + \pi^\mu$. Consider the interval $s \in [0, 1]$ and $X^\mu(s)$ such that

$$X^\mu(0) = \chi^\mu, \quad \partial_s X^\mu|_{s=0} = \xi^\mu, \quad X^\mu(1) = \chi^\mu + \pi^\mu, \quad (3.1)$$

which in addition satisfies the geodesic equation

$$\partial_s^2 X^\mu + \Gamma_{\nu\kappa}^\mu \partial_s X^\nu \partial_s X^\kappa = 0, \quad (3.2)$$

where $\Gamma_{\nu\kappa}^\mu$ is the standard Levi-Civita connection, built from a metric. It turns out that the covariant Lagrangian is given by a Taylor expansion in the parameter s , as

$$\mathcal{L}(s=1) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_s^n \mathcal{L}|_{s=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \nabla_s^n \mathcal{L}|_{s=0}, \quad (3.3)$$

where ∇_s is the covariant derivative along to the curve $X^\mu(s)$. For later convenience, we present the formulae

$$\begin{aligned} \nabla_s \partial_s X^\mu &= 0, \quad [\nabla_s, \nabla_a] X^\mu = R_{\kappa\nu;\rho}^\mu X^\rho \partial_s X^\kappa \partial_a X^\nu, \\ \nabla_s \partial_a X^\mu &= \nabla_a \partial_s X^\mu = \partial_a \partial_s X^\mu + \Gamma_{\kappa\lambda}^\mu \partial_a X^\kappa \partial_s X^\lambda, \end{aligned} \quad (3.4)$$

where ∇_s denotes the worldsheet coordinates from which the fields X^μ depend, in our case $\alpha = 0, 1$. Now, we are in position to apply this method directly to (2.1) after

we choose what our metric in this expansion would be. The two possibilities are, the symmetric part of $C_{\mu\nu}$, i.e. $S_{\mu\nu}$ and $M_{\mu\nu}$.

To calculate the one-loop effective action of (2.1) we have to perform a path integral over the ξ fluctuations and therefore we need to calculate the Weyl and Lorentz anomalies of the divergent contributions of the second order expansion of this action. These come schematically from the ξ^2 and ξ^4 contractions. In particular, these divergences originate from single contractions in $\langle \xi\xi \rangle$ and double contractions in $\langle \xi\partial\xi\xi\partial\xi \rangle$. Due to the fact that the candidate metrics $S_{\mu\nu}$ and $M_{\mu\nu}$ depend on the background fields χ^μ , it is convenient to go to the tangent frame, defined by the vielbeins of our metric. For our purposes we take $S_{\mu\nu}$ to be our metric.

3.2 Background field expansion

We consider the covariant expansions the two terms in (2.1) separately and label them as \mathcal{S}_{WZW} and \mathcal{S}_{NL} , respectively.

3.2.1 The WZW action

The WZW part of the Lagrangian (2.1), can be written as

$$\mathcal{S}_{\text{WZW}} = \frac{1}{2} \int d\tau d\sigma C_{\mu\nu} \partial_0 X^\mu \partial_1 X^\nu = \frac{1}{2} \int d\tau d\sigma (S_{\mu\nu} + B_{\mu\nu}) \partial_0 X^\mu \partial_1 X^\nu, \quad (3.5)$$

where $S_{\mu\nu}$ and $B_{\mu\nu}$ are the symmetric and antisymmetric parts of $C_{\mu\nu}$. The first derivative of the action with respect to s after an integration by parts in the term involving the two form potential, has Lagrangian density given by

$$\frac{1}{2} (S_{\mu\nu} \nabla_0 \partial_s X^\mu \partial_1 X^\nu + \partial_0 X^\mu \nabla_1 \partial_s X^\nu + H_{\lambda\mu\nu} \partial_s X^\lambda \partial_0 X^\mu \partial_1 X^\nu) , \quad (3.6)$$

where $H_{\kappa\lambda\mu} = \partial_{[\kappa} B_{\lambda\mu]}$ and note we used the fact that we have a Levi-Civita connection. Using that we may compute the second derivative of the action with respect to s and therefore after setting $s = 0$ the second order term in the expansion (3.3). We eventually find that

$$\begin{aligned} \frac{1}{2} \mathcal{S}_{\text{WZW}}^{(2)} \Big|_{s=0} &= \frac{1}{2} \int d\tau d\sigma \left[S_{\mu\nu} \nabla_0 \xi^\mu \nabla_1 \xi^\nu + (R_{\kappa\mu;\nu\lambda} + \frac{1}{2} \nabla_\kappa H_{\lambda\mu\nu}) \xi^\kappa \xi^\lambda \partial_0 \chi^\mu \partial_1 \chi^\nu \right. \\ &\quad \left. + \frac{1}{2} H_{\lambda\mu\nu} \xi^\lambda (\nabla_0 \xi^\mu \partial_1 \chi^\nu + \partial_0 \chi^\mu \nabla_1 \xi^\nu) \right] . \end{aligned} \quad (3.7)$$

Equivalently, this can be written in terms of the generalized Riemann tensor that uses $\hat{\Gamma}_{\nu\lambda}^\mu$ as a connection. Recalling that⁴

$$\hat{R}_{\mu\nu;\rho\lambda} = R_{\mu\nu;\rho\lambda} - \frac{1}{2}\nabla_{[\nu}H_{\mu]\rho\lambda} + \frac{1}{4}H_{\rho[\mu}{}^\alpha H_{\nu]\lambda\alpha} , \quad (3.8)$$

we arrive at

$$\frac{1}{2}\mathcal{S}_{WZW}^{(2)} = \frac{1}{2}\int d\tau d\sigma \left[S_{\mu\nu}\hat{\nabla}_0\xi^\mu\hat{\nabla}_1\xi^\nu + \hat{R}_{\kappa\mu;\nu\lambda}\xi^\kappa\xi^\lambda\partial_0\chi^\mu\partial_1\chi^\nu \right] , \quad (3.9)$$

where we have introduced the generalized covariant derivatives

$$\begin{aligned} \hat{\nabla}_0\xi^\nu &= \nabla_0\xi^\nu + \frac{1}{2}H_{\kappa}{}^\nu{}_\lambda\partial_0\chi^\kappa\xi^\lambda , \\ \hat{\nabla}_1\xi^\nu &= \nabla_1\xi^\nu - \frac{1}{2}H_{\kappa}{}^\nu{}_\lambda\partial_1\chi^\kappa\xi^\lambda , \end{aligned} \quad (3.10)$$

with

$$\nabla_a\xi^\mu = \partial_a\xi^\mu + \Gamma_{\nu\lambda}^\mu\partial_a\chi^\nu\xi^\lambda . \quad (3.11)$$

Note that the sign swap in the second terms in (3.10) above indicates that this is not simply the covariant derivative with generalized connection pulled back to the world sheet in a naive way.

Using the special form of our background and in particular (2.46) and (2.47) we find from (3.10) that

$$\begin{aligned} \hat{\nabla}_0\xi^\rho &= \nabla_0\xi^\rho + \frac{1}{2}H_{\lambda}{}^\rho{}_\mu\partial_0\chi^\lambda\xi^\mu \\ &= L_A^\rho(\nabla_0\xi^A - \frac{1}{2}f_B{}^A{}_CL_\nu^B\partial_0\chi^\nu\xi^C) \\ &= L_A^\rho(\partial_0\xi^A + \omega_\nu{}^A{}_C\partial_0\chi^\nu\xi^C - \frac{1}{2}f_B{}^A{}_CL_\nu^B\partial_0\chi^\nu\xi^C) \\ &= L_A^\rho\partial_0\xi^A , \end{aligned} \quad (3.12)$$

where $\xi^\rho = L_A^\rho\xi^A$ and

$$\begin{aligned} \hat{\nabla}_1\xi^\rho &= \nabla_1\xi^\rho - \frac{1}{2}H_{\lambda}{}^\rho{}_\mu\partial_1\chi^\lambda\xi^\mu \\ &= L_A^\rho(\nabla_1\xi^A + \frac{1}{2}f_B{}^A{}_CL_\nu^B\partial_1\chi^\nu\xi^C) \\ &= L_A^\rho(\partial_1\xi^A + \omega_\nu{}^A{}_C\partial_1\chi^\nu\xi^C + \frac{1}{2}f_B{}^A{}_CL_\nu^B\partial_1\chi^\nu\xi^C) \\ &= L_A^\rho(\partial_1\xi^A + f^A{}_{BC}\xi^BL_\lambda^C\partial_1\chi^\lambda) . \end{aligned} \quad (3.13)$$

⁴In our conventions we antisymmetrize indices as $[ab] = ab - ba$.

Note that these expressions depend crucially on the relative sign between the torsion and the spin connection. In the analysis of a standard WZW either sign choice is consistent and leads to the same ultimate result. However, in our case of non-Lorentz invariant theories we found that we had to specify opposite signs in (2.36) to ensure on shell invariance.

Finally, from Eq.(3.9), (3.13) and (3.14) we obtain that

$$\frac{1}{2} \mathcal{S}_{\text{WZW}}^{(2)} = \frac{1}{2} \int d\tau d\sigma [\eta_{AB} \partial_0 \xi^A \partial_1 \xi^B - f_{ABC} \xi^A \partial_0 \xi^B L_k^C \partial_1 \chi^k] . \quad (3.14)$$

Clearly, upon exponentiation and wick contraction there is no divergence. This can be easily seen from (3.14) since, when we only have the WZW model kinetic term the propagator is irregular in the 01 directions and proportional to η^{AB} . Thus, the $\xi \partial_0 \xi$ term vanishes, whereas the $(\xi \partial_0 \xi)^2$ is finite. This is the familiar statement that the WZW model is not renormalized and is conformal. However, as we will now come onto, when we also consider the interaction term \mathcal{S}_{NL} we modify the form of the propagators and the exponentiation of (3.14) does provide an important contribution to both the Weyl divergence and the Lorentz anomaly.

3.2.2 The NL action

The second term of the Lagrangian (2.1), is

$$\mathcal{S}_{\text{NL}} = \frac{1}{2} \int d\tau d\sigma M_{\mu\nu} \partial_1 X^\mu \partial_1 X^\nu . \quad (3.15)$$

The first derivative of the action with respect to s has Lagrangian density given by

$$\frac{1}{2} (\nabla_\kappa M_{\mu\nu} \partial_s X^\kappa \partial_1 X^\mu \partial_1 X^\nu + 2 M_{\mu\nu} \nabla_1 \partial_s X^\mu \partial_1 X^\nu) . \quad (3.16)$$

Similarly, to (3.7) we find that the second order term in the expansion (3.3) is

$$\begin{aligned} \frac{1}{2} \mathcal{S}_{\text{NL}}^{(2)}|_{s=0} &= \frac{1}{2} \int d\tau d\sigma \left[M_{\mu\nu} \nabla_1 \xi^\mu \nabla_1 \xi^\nu + (R_{\kappa\mu;\nu\lambda} + \frac{1}{2} \nabla_\kappa \nabla_\lambda M_{\mu\nu}) \xi^\kappa \xi^\lambda \partial_1 \chi^\mu \partial_1 \chi^\nu \right. \\ &\quad \left. + 2 \nabla_k M_{\mu\nu} \xi^\kappa \nabla_1 \xi^\mu \partial_1 \chi^\nu \right] . \end{aligned} \quad (3.17)$$

Expanding this using $M_{\mu\nu} = -L_\mu^A L_\nu^B R_{AB}$ and a lengthy procedure we obtain an analogous to (3.14) action for the fluctuations.

3.3 Calculation of the effective action

The action for the fluctuations of the complete action (2.1) can be written as a sum of a kinetic and interacting terms for the ξ^A 's, as

$$S^{(2)} = S_{\text{kin}} + S_{\text{int}} , \quad (3.18)$$

where

$$S_{\text{kin}} = \frac{1}{2} \int d\sigma d\tau \left(\eta_{AB} \partial_0 \xi^A \partial_1 \xi^B - R_{AB} \partial_1 \xi^A \partial_1 \xi^B \right) , \quad (3.19)$$

and

$$S_{\text{int}} = \frac{1}{2} \int d\sigma d\tau \left(I_{AB} \xi^A \xi^B + J_{AB} \xi^A \partial_1 \xi^B + K_{AB} \xi^A \partial_0 \xi^B \right) , \quad (3.20)$$

with

$$\begin{aligned} I_{AB} &= -\frac{1}{2} \partial_1 \chi^\mu \partial_1 \chi^\nu L_\mu^C L_\nu^D \left[f_{AC}{}^E f_{BD}{}^F R_{EF} + (2f_{AF}{}^E R_{EC} + f_{AC}{}^E R_{EF}) f_{BD}{}^F \right] , \\ J_{AB} &= (f_{BA}{}^C R_{CE} + 2f_{EA}{}^C R_{CB}) L_\mu^E \partial_1 \chi^\mu , \\ K_{AB} &= -f_{ABC} L_\mu^C \partial_1 \chi^\mu . \end{aligned} \quad (3.21)$$

We note that the terms labeled by I_{AB} and J_{AB} come from the $\mathcal{S}_{\text{NL}}^{(2)}$ action, while the term K_{AB} comes from the $\mathcal{S}_{\text{WZW}}^{(2)}$ action.

The effective action $S_{\text{eff}}(\chi)$ at one loop, it is given in terms of the interacting Lagrangian as

$$e^{iS_{\text{eff}}(\chi)} = \int [D\xi] e^{iS^{(2)}[\chi, \xi]} \implies S_{\text{eff}}(\chi) = \langle S_{\text{int}} \rangle + \frac{i}{2} \langle S_{\text{int}}^2 \rangle . \quad (3.22)$$

Substituting (3.18) in (3.22) we find that the effective Lagrangian is given by

$$\begin{aligned} S_{\text{eff}}(\chi) &= \frac{1}{2} \int d\sigma d\tau (q_1 + q_2 + q_3 + q_4) \\ &= \frac{1}{2} \int d\sigma d\tau \left(I_{AB} \langle \xi^A \xi^B \rangle + \frac{i}{4} J_{AB} J_{CD} \langle \xi^A \partial_1 \xi^B \xi^C \partial_1 \xi^D \rangle \right. \\ &\quad \left. + \frac{i}{4} K_{AB} K_{CD} \langle \xi^A \partial_0 \xi^B \xi^C \partial_0 \xi^D \rangle + \right. \\ &\quad \left. + \frac{i}{4} (J_{AB} K_{CD} + K_{AB} J_{CD}) \langle \xi^A \partial_1 \xi^B \xi^C \partial_0 \xi^D \rangle \right) , \end{aligned} \quad (3.23)$$

where $q_i, i = 1, 2, 3, 4$ are respectively the first, second, third and fourth term in the above expression. As for the ξ^2 contractions and $(\xi \partial \xi)^2$, these can be found from the S_{kin} and

were computed in detail in the appendix of [16] and will not be repeated here. The end results are

$$\begin{aligned}
\langle \xi^A \xi^B \rangle &= R^{AB} \Delta(0) + \eta^{AB} \Theta(0) , \\
i \langle \xi^A \partial_1 \xi^B \xi^C \partial_1 \xi^D \rangle &\simeq \frac{1}{2} (R^{A[C} R^{D]B} - \eta^{A[C} \eta^{D]B}) \Delta(0) , \\
i \langle \xi^A \partial_0 \xi^B \xi^C \partial_0 \xi^D \rangle &\simeq -\frac{1}{2} (R^{A[C} R^{D]B} + 3\eta^{A[C} \eta^{D]B}) \Delta(0) \\
&\quad - (R^{A[C} \eta^{D]B} + \eta^{A[C} R^{D]B}) \Theta(0) , \\
\langle \xi^A \partial_0 \xi^B \xi^C \partial_1 \xi^D \rangle &\simeq -\frac{1}{2} (R^{A[C} \eta^{D]B} + \eta^{A[C} R^{D]B}) \Delta(0) - \eta^{A[C} \eta^{D]B} \Theta(0) ,
\end{aligned} \tag{3.24}$$

where $\Delta(0)$ and $\Theta(0)$ are the Weyl and Lorentz anomalies, respectively and \simeq denotes the fact that we have kept only the terms relevant to the Lorentz and the Weyl divergence.⁵

3.3.1 Weyl anomaly

We shall now compute the Weyl anomaly of the effective action by plugging (3.21) and (3.24) in (3.23). The result can be written as

$$\begin{aligned}
q_1^W &= -\frac{1}{2} \partial_1 \chi^\mu \partial_1 \chi^\nu L_\mu^C L_\nu^D (f_{AC}{}^E f_{BD}{}^F R_{EF} R^{AB} + \\
&\quad (2f_{AF}{}^E R_{EC} + f_{AC}{}^E R_{EF}) f_{BD}{}^F R^{AB}) , \\
q_2^W &= \frac{1}{4} f_{AB}{}^C f_{DE}{}^F R_{CE} R_{FK} (R^{AD} R^{BE} - \eta^{AD} \eta^{BE}) \partial_1 \chi^\mu \partial_1 \chi^\nu L_\mu^E L_\nu^K , \\
q_3^W &= -\frac{1}{4} (f_{ABC} f_{DEF} R^{AD} R^{BE} + 3f_{ABC} f^{AB}{}_F) \partial_1 \chi^\mu \partial_1 \chi^\nu L_\mu^C L_\nu^F , \\
q_4^W &= (f_{ABC} f^{AB}{}_D + f_{CE}{}^F R_{FB} f_{DA}{}^B R^{EA} + f_{ABC} f^{AF}{}_E R_{FD} R^{EB}) \partial_1 \chi^\mu \partial_1 \chi^\nu L_\mu^C L_\nu^D ,
\end{aligned} \tag{3.25}$$

where we note that we have made heavy use of the compatibility condition (2.24) in the calculation of q_2 . Adding up the contributions we find that the Weyl anomaly reads

$$\begin{aligned}
\text{Weyl} &= \frac{1}{4} f_{AB}{}^C f_{DE}{}^F (R_{CK} R_{FH} R^{AD} R^{BE} - R_{CK} R_{FH} \eta^{AD} \eta^{BE} + \\
&\quad \eta_{CK} \eta_{FH} \eta^{AD} \eta^{BE} - \eta_{CK} \eta_{FH} R^{AD} R^{BE}) L_\mu^K L_\nu^H \partial_1 \chi^\mu \partial_1 \chi^\nu \\
&= \frac{1}{4} (R_{AC} R_{BF} - \eta_{AC} \eta_{BF}) (R^{KD} R^{HE} - \eta^{KD} \eta^{HE}) f_{KH}{}^C f_{DE}{}^F \times \\
&\quad L_\mu^A L_\nu^B \partial_1 \chi^\mu \partial_1 \chi^\nu .
\end{aligned} \tag{3.26}$$

⁵In these expressions $\langle \xi^A \xi^B \rangle$ is really short hand for $\int d^2 \sigma' \langle \xi^A(\sigma) \xi^B(\sigma') \rangle$.

However, the zeroth order in the expansion of \mathcal{S}_{NL} has the Lagrangian density

$$M_{\mu\nu}\partial_1\chi^\mu\partial_1\chi^\nu = -R_{AB}L_\mu^AL_\nu^B\partial_1\chi^\mu\partial_1\chi^\nu. \quad (3.27)$$

Thus, since the Weyl anomaly counter terms can be absorbed as a redefinition of the R_{AB} we find that the action (2.1) is renormalizable at one loop. Moreover the RG flow of R_{AB} can be directly read off from (3.26) and it is equal to

$$\boxed{\frac{dR_{AB}}{dt} = \frac{1}{4}(R_{AC}R_{BF} - \eta_{AC}\eta_{BF})(R^{KD}R^{HE} - \eta^{KD}\eta^{HE})f_{KH}{}^C f_{DE}{}^F}, \quad (3.28)$$

where $t = \ln \mu$, with μ being the energy scale. This is a quite simple formula and constitutes one of the main results of present paper. One can readily check that it is compatible with the condition (2.24).⁶

3.3.2 Lorentz anomaly

We shall now compute the Lorentz anomaly of the effective action by plugging (3.21) and (3.24) in (3.23). The result can be written as

$$\begin{aligned} q_1^L &= -\frac{1}{2}\partial_1\chi^\mu\partial_1\chi^\nu L_\mu^CL_\nu^D (f_{AC}{}^E f_{BD}{}^F R_{EF}\eta^{AB} + \\ &\quad (2f_{AF}{}^E R_{EC} + f_{AC}{}^E R_{EF})f_{BD}{}^F \eta^{AB}) , \\ q_2^L &= 0 , \\ q_3^L &= f_{ABC}f_{DE}{}^B R^{AD}L_\mu^CL_\nu^E \partial_1\chi^\mu\partial_1\chi^\nu , \\ q_4^L &= (-f_{AB}{}^C R_{CE} + 2f_{EA}{}^C R_{CB})f^{AB}{}_D L_\mu^E L_\nu^D \partial_1\chi^\mu\partial_1\chi^\nu . \end{aligned} \quad (3.29)$$

Adding up the contributions we find that the Lorentz anomaly is zero. Thus, the system is Lorentz invariant at one loop and this represents an important and non-trivial demonstration of consistency.

4 Application to Poisson–Lie T-duality

Let's concentrate to the case where the group D providing the symmetry structure in our construction is the Drinfeld double of two groups \mathcal{G} and $\tilde{\mathcal{G}}$ of equal dimension d_g . Splitting

⁶It is interesting and quite curious that the right hand side of (3.28) appeared before in a study of generalizations of WZW-type conformal models [25].

the index $A = (a, \tilde{a})$ and using the convention $T^{\tilde{a}} = \tilde{T}_a$, the commutation relations can be written as

$$\begin{aligned} [T_a, T_b] &= i f_{ab}{}^c T_c , \\ [\tilde{T}^a, \tilde{T}^b] &= i \tilde{f}^{ab}{}_c \tilde{T}^c , \\ [T_a, \tilde{T}^b] &= i \tilde{f}^{bc}{}_a T_c - i f_{ac}{}^b \tilde{T}^c . \end{aligned} \tag{4.1}$$

It turns out that starting from the action (2.55) (with no spectators) and parameterizing the group element $G \in D$ in two inequivalent ways, namely as $G = \tilde{h}g$ and $G = h\tilde{g}$, where $h, g \in \mathcal{G}$ and $\tilde{h}, \tilde{g} \in \tilde{\mathcal{G}}$, we may solve some constraint type equations [12] for the fields in \tilde{h} or those in h and obtain Lorentz-invariant σ -model actions of the form

$$S = \frac{1}{2} \int Q_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu , \tag{4.2}$$

for the remaining fields in g or in \tilde{g} , respectively.

In the absence of spectator fields, these dual two-dimensional σ -model actions are [7]

$$S = \frac{1}{2} \int E_{ab} L_\mu^a L_\nu^b \partial_+ X^\mu \partial_- X^\nu , \quad E = (M - \Pi)^{-1} , \tag{4.3}$$

and

$$\tilde{S} = \frac{1}{2} \int \tilde{E}^{ab} \tilde{L}_{a\mu} \tilde{L}_{b\nu} \partial_+ \tilde{X}^\mu \partial_- \tilde{X}^\nu , \quad \tilde{E} = (M^{-1} - \tilde{\Pi})^{-1} , \tag{4.4}$$

where M is an arbitrary square matrix of $\dim(\mathcal{G})$ and Π_{ab} ($\tilde{\Pi}^{ab}$) is an X^μ (\tilde{X}^μ)-dependent matrix which subtly encodes the group structure of the Double, but whose details are not relevant in our considerations.

The one-loop beta-functions are expressed as

$$\frac{da^i}{dt} = \frac{1}{\pi} a_1^i , \tag{4.5}$$

where the a_1^i 's are chosen such that

$$\frac{1}{2} \hat{R}_{\mu\nu} = \partial_{a^i} Q_{\mu\nu}^+ a_1^i , \tag{4.6}$$

where in our case the couplings are the entries of the matrix M in (4.3) and (4.4). Also, $\hat{R}_{\mu\nu}$ are the components of the generalized Ricci tensor defined with the connection $\hat{\Gamma}_{\nu\rho}^\mu$ that includes the torsion. The counter-terms were computed in [21, 26, 27] and we have also omitted possible field renormalizations as they are not needed in our case.

To present the result for the beta function equations we recall the notation introduced in [15]. We first define the matrices

$$A^{ab}{}_c = \tilde{f}^{ab}{}_c - f_{cd}{}^a M^{db}{}_c, \quad B^{ab}{}_c = \tilde{f}^{ab}{}_c + M^{ad} f_{dc}{}^b, \quad (4.7)$$

as well as their duals

$$\tilde{A}_{ab}{}^c = f_{ab}{}^c - \tilde{f}^{cd}{}_a (M^{-1})_{db}, \quad \tilde{B}_{ab}{}^c = f_{ab}{}^c + (M^{-1})_{ad} \tilde{f}^{dc}{}_b, \quad (4.8)$$

Using these we construct also

$$\begin{aligned} L^{ab}{}_c &= \frac{1}{2} (M_s^{-1})_{cd} (B^{ab}{}_e M^{ed} + A^{db}{}_e M^{ae} - A^{ad}{}_e M^{eb}) , \\ R^{ab}{}_c &= \frac{1}{2} (M_s^{-1})_{cd} (A^{ab}{}_e M^{de} + B^{ad}{}_e M^{eb} - B^{db}{}_e M^{ae}) \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \tilde{L}_{ab}{}^c &= \frac{1}{2} (\tilde{M}_s^{-1})^{cd} (\tilde{B}_{ab}{}^e (M^{-1})_{ed} + \tilde{A}_{db}{}^e (M^{-1})_{ae} - \tilde{A}_{ad}{}^e (M^{-1})_{eb}) , \\ \tilde{R}_{ab}{}^c &= \frac{1}{2} (\tilde{M}_s^{-1})^{cd} (\tilde{A}_{ab}{}^e (M^{-1})_{de} + \tilde{B}_{ad}{}^e (M^{-1})_{eb} - \tilde{B}_{db}{}^e (M^{-1})_{ae}) , \end{aligned} \quad (4.10)$$

where

$$M_s = \frac{1}{2} (M + M^T), \quad \tilde{M}_s = \frac{1}{2} [(M^{-1}) + (M^{-1})^T]. \quad (4.11)$$

The one-loop RG flow system of equations corresponding to (4.3) can be read from [14]. In the notation of [15] it reads

$$\frac{dM^{ab}}{dt} = R^{ac}{}_d L^{db}{}_c. \quad (4.12)$$

Similarly, for its dual (4.4) we have

$$\frac{d(M^{-1})_{ab}}{dt} = \tilde{R}_{ac}{}^d \tilde{L}_{db}{}^c, \quad (4.13)$$

up to a constant overall factor absorbed into a redefinition of t .

Developing certain identities among the various quantities defined above it has been shown in [15] that the two systems (4.12) and (4.13) are in fact equivalent. Therefore, at one-loop in perturbation theory, general σ -models related by Poisson–Lie T-duality are equivalent under the renormalization group flow a fact that seems to be related to the fact that the σ -model actions (4.3) and (4.4) are on fact canonically equivalent in phase space [8, 9].

The equivalence of the systems (4.12) and (4.13) does not necessarily mean that the beta functions one would have computed for the original action (2.1) using (3.28) would have been the same. The reason is that as we have mentioned above certain constraints were solved in order to obtain (4.3) and (4.4) and quantization before and after solving them might not be commuting operations. In order to check this issue we need to conveniently parametrize the matrix R_{AB} introduced in (2.22) in the case that the group is a Driinfeld double. First we define an inner product as

$$\langle T_A | T_B \rangle = \eta_{AB} = \begin{pmatrix} \mathbf{0} & \mathbb{I}_{d_g \times d_g} \\ \mathbb{I}_{d_g \times d_g} & \mathbf{0} \end{pmatrix} \quad (4.14)$$

and then the basis

$$R_a^+ = T_a + M_{ab}^{-1} \tilde{T}^b, \quad R_a^- = T_a - M_{ba}^{-1} \tilde{T}^b. \quad (4.15)$$

We easily verify that

$$\langle R_a^\pm | R_b^\pm \rangle = \pm 2 \tilde{M}_s, \quad \langle R_a^\pm | R_b^\mp \rangle = 0, \quad (4.16)$$

as well as completeness

$$\frac{1}{2} |R_a^+\rangle (\tilde{M}_s^{-1})^{ab} \langle R_b^+| - \frac{1}{2} |R_a^-\rangle (\tilde{M}_s^{-1})^{ab} \langle R_b^-| = \mathbb{I}. \quad (4.17)$$

Then we define the operator

$$R = \frac{1}{2} |R_a^+\rangle (\tilde{M}_s^{-1})^{ab} \langle R_b^+| + \frac{1}{2} |R_a^-\rangle (\tilde{M}_s^{-1})^{ab} \langle R_b^-|, \quad (4.18)$$

whose inner products

$$R_{AB} = \langle T_A | R | T_B \rangle, \quad (4.19)$$

provide the necessary matrix elements for the computation in (3.28). Explicitly, in a block diagonal form, they are given by

$$R_{AB} = \begin{pmatrix} \tilde{M}_s - B \tilde{M}_s^{-1} B & -B \tilde{M}_s^{-1} \\ \tilde{M}_s^{-1} B & \tilde{M}_s^{-1} \end{pmatrix}, \quad (4.20)$$

where $B = \frac{1}{2} [M^{-1} - (M^{-1})^T]$.

Before we present some examples we cannot refrain from noting the complexity of the systems (4.12) and (4.13) as compared to the manifestly duality invariant system (3.28).

4.1 Examples

We have not attempted to prove in general that in the case of Drinfeld doubles the RG flow system of equations (3.28) reduce to (4.12) (equivalently (4.13)). Instead, we will present three non-trivial examples in which we will explicitly compute the beta functions equations using (3.28) and recover the same equations that were previously computed using (4.12).

4.1.1 Semi-Abelian Doubles

As a first example we consider the case when the commutator relations for the double are given as

$$\begin{aligned} [T_a, T_b] &= if_{ab}{}^c T_c , \\ [\tilde{T}^a, \tilde{T}^b] &= 0 , \\ [T_a, \tilde{T}^b] &= -if_{ac}{}^b \tilde{T}^c . \end{aligned} \tag{4.21}$$

This is known as the semi-abelian double since $\tilde{\mathcal{G}} = U(1)^d$ and coincides with the regular notion non-abelian T-duality. We leave the group \mathcal{G} general but to keep the problem simple we consider a point in the coupling space where

$$M^{ab} = \kappa \delta^{ab} . \tag{4.22}$$

Then from (4.9) we find that $R^{ab}{}_c = -L^{ab}{}_c = \frac{\kappa}{2} f^{ab}{}_c$. Then, from the general RG equations (4.12) we find the classical result

$$\frac{d\kappa}{dt} = -\frac{C_2}{4} \kappa^2 , \tag{4.23}$$

where C_2 is the quadratic Casimir in the adjoint representation. In the duality invariant expressions we have that $R_{AB} = \text{diag}(\frac{1}{\kappa} \delta_{ab}, \kappa \delta^{ab})$ and can simply calculate the RG equation for κ using (3.28). We find that the above result is recovered.

4.1.2 A six-dimensional Drinfeld double

In this example we use a six-dimensional Drinfeld double based on the three-dimensional Lie algebras, IX for \mathcal{G} and V for $\tilde{\mathcal{G}}$ in the Bianchi classification. The corresponding generators are T_a and \tilde{T}^a , where $a = 1, 2, 3$. It is convenient to split the index $a = (3, \alpha)$,

with $\alpha = 1, 2$. The non-vanishing commutation relations are

$$\begin{aligned} [T_a, T_b] &= i\epsilon_{abc}T_c, & [\tilde{T}_3, \tilde{T}_\alpha] &= i\tilde{T}_\alpha, \\ [T_\alpha, \tilde{T}_\beta] &= i\epsilon_{\alpha\beta}\tilde{T}_3 - i\delta_{\alpha\beta}T_3, & [T_3, \tilde{T}_\alpha] &= i\epsilon_{\alpha\beta}\tilde{T}_\beta, & [\tilde{T}_3, T_\alpha] &= i\epsilon_{\alpha\beta}\tilde{T}_\beta - iT_\alpha \end{aligned} \quad (4.24)$$

where $\delta_{\alpha\beta}, \epsilon_{\alpha\beta}$ are the Kronecker delta and the antisymmetric symbol in two dimensions and ϵ_{abc} is the totally antisymmetric tensor in three dimensions, $\epsilon_{123} = 1$. From these one can read off the structure constants.

As for the matrix R_{AB} , as it was previously shown it can be constructed in terms the symmetric and antisymmetric part of M^{-1} , where M is taken for simplicity to be

$$M = \begin{pmatrix} a & 1-b & 0 \\ b-1 & a & 0 \\ 0 & 0 & \frac{a}{1+g} \end{pmatrix}. \quad (4.25)$$

Then from (4.20)

$$R = \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 & \frac{1-b}{a} & 0 \\ 0 & \frac{1}{a} & 0 & \frac{b-1}{a} & 0 & 0 \\ 0 & 0 & \frac{1+g}{a} & 0 & 0 & 0 \\ 0 & \frac{b-1}{a} & 0 & \frac{a^2+(b-1)^2}{a} & 0 & 0 \\ \frac{1-b}{a} & 0 & 0 & 0 & \frac{a^2+(b-1)^2}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{a}{1+g} \end{pmatrix}. \quad (4.26)$$

Whereas, the matrix η_{AB} is given by (4.14) with $d_g = 3$. Plugging the structure constants and (4.26) into (3.28) we find

$$\begin{aligned} \frac{da}{dt} &= \frac{1+a^2-b^2}{2a^2}((g-1)a^2 + (g+1)(b^2-1)), \\ \frac{db}{dt} &= \frac{b}{a}(a^2(g-1) + (g+1)(b^2-1)), \\ \frac{dg}{dt} &= \frac{1+g}{a}(g(1+a^2) + (g+2)b^2), \end{aligned} \quad (4.27)$$

which is precisely the same system one derives using (4.12) or (4.13) [20, 15].

4.1.3 A sixteen-dimensional Drinfeld double

In this final example we study a sixteen-dimensional Drinfeld double group based on an $SU(3)$ group with generators T_a and an abelian eight-dimensional group with generators

\tilde{T}^a , where $a = 1, 2, \dots, 8$. For the $SU(3)$ group we use the structure constants in the Gell-Mann basis (see for instance, eq.(5.2) of [28])

$$f^{12}_3 = 2, \quad f^{14}_7 = -f^{15}_6 = f^{24}_6 = f^{25}_7 = f^{34}_5 = -f^{36}_7 = 1, \quad f^{45}_8 = f^{67}_8 = \sqrt{3} \quad (4.28)$$

where the rest are obtained by antisymmetrization in the three indices. Note that there is an $SU(2)$ subgroup generated by T^i , $i = 1, 2, 3$. The non-vanishing structure constants are read by comparing the first and third equation of (4.1). In particular, we find that $f^{a\bar{b}}_{\bar{c}} = f^{ab}_c$ and antisymmetry in the first two indices is understood. As for the matrix R_{AB} , it is built in terms of the symmetric and antisymmetric part of M^{-1} , where M is taken for simplicity to be a diagonal eight-dimensional matrix

$$M = \left(\frac{1}{g} \mathbb{I}_{3 \times 3} \right) \oplus (a \mathbb{I}_{4 \times 4}) \oplus b. \quad (4.29)$$

The result is a diagonal sixteen-dimensional matrix given by

$$R = (g \mathbb{I}_{3 \times 3}) \oplus \left(\frac{1}{a} \mathbb{I}_{4 \times 4} \right) \oplus \frac{1}{b} \oplus \left(\frac{1}{g} \mathbb{I}_{3 \times 3} \right) \oplus (a \mathbb{I}_{4 \times 4}) \oplus b. \quad (4.30)$$

Whereas the matrix η_{AB} is given by (4.14) with $d_G = 8$. Plugging the structure constants, (4.14), (4.30) into (3.28) we find

$$\frac{da}{dt} = \frac{3a^2}{2} \frac{a(bg + 1) - 4b}{b}, \quad \frac{db}{dt} = -3a^2, \quad \frac{dg}{dt} = g^2 a^2 + 2, \quad (4.31)$$

which were precisely the expressions in eq.(5.9) of [15].

5 Conclusions

We showed that demanding on-shell Lorentz invariance highly constrains the structure of a general Lorentz non-invariant action. The resulting theories have an underlying group structure and consist of a WZW term together with some interaction term. In the case that the group is a Drinfeld Double these theories are Poisson–Lie T-duality invariant σ -models. By using a background field method we calculated the one-loop effective action of these models and found that they were renormalizable and that a possible quantum Lorentz anomaly vanished. This is an important consistency condition of such models. We also obtained the renormalization group equations for the couplings of the interaction term in our model. For the Poisson–Lie T-duality invariant theories this provides a duality invariant description of the RG equations, a key motivator for this work. We also verified that for specific examples of the Drinfeld Double that these

duality invariant RG equations agreed with those obtained from either of the T-dual standard σ -models. A very interesting open problem is to prove this agreement in full generality, although we believe that the non-trivial examples that we have given leave not much doubt that this will be the case.

An observation, that hints at the utility of a duality invariant framework, is that duality invariant RG equations can be computed using simple contractions of structure constants with constant matrices. When dealing with the standard σ -model one obtains these equations only by calculating the generalized curvature of what are, in general, extremely complicated target space backgrounds.

Note added: Towards the completion of typing the present paper, the work of [29] appeared where similar issues are discussed, albeit from a different point of view and motivation. The result (3.28) has also been derived in that work as well. We acknowledge that the authors communicated to us their results a few days before submission.

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